Appendix

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Contents

1 Basic properties of CP_2

1.1 CP_2 as a manifold

CP_2, the complex projective space the complex projective space of two complex dimensions, is obtained by identifying the points of complex 3-space C^3 under the projective equivalence

$$(z^1, z^2, z^3) \equiv \lambda(z^1, z^2, z^3) \ . \tag{1}$$

Here λ is any non-zero complex number. Note that CP_2 can also regarded as the coset space $SU(3)/U(2)$. The pair z^i/z^j for fixed j and $z^i \neq 0$ defines a complex coordinate chart for CP_2. As j runs from 1 to 3 one obtains an atlas of three oordinate charts covering CP_2, the charts being holomorphically related to each other (e.g. CP_2 is a complex manifold). The points $z^3 \neq 0$ form a subset of CP_2 homoeomorphic to R^4 and the points with $z^3 = 0$ a set homeomorphic to S^2. Therefore CP_2 is obtained by "adding the 2-sphere at infinity to R^4".

Besides the standard complex coordinates $\xi^i = z^i/z^3$, $i = 1, 2$ the coordinates of Eguchi and Freund [2] will be used and their relation to the complex coordinates is given by

$$
\begin{aligned}
\xi^1 &= z + it \ , \\
\xi^2 &= x + iy \ .
\end{aligned}
\tag{2}
$$

These are related to the "spherical coordinates" via the equations

$$
\begin{aligned}
\xi^1 &= r\exp(i\frac{(\Psi + \Phi)}{2})\cos(\frac{\Theta}{2}) \ , \\
\xi^2 &= r\exp(i\frac{(\Psi - \Phi)}{2})\sin(\frac{\Theta}{2}) \ .
\end{aligned}
\tag{3}
$$

The ranges of the variables r, Θ, Φ, Ψ are $[0, \infty], [0, \pi], [0, 4\pi], [0, 2\pi]$ respectively.

Considered as a real four-manifold CP_2 is compact and simply connected, with Euler number Euler number 3, Pontryagin number 3 and second $b = 1$.

1.2 Metric and Kähler structures of CP_2

In order to obtain a natural metric for CP_2, observe that CP_2 can be thought of as a set of the orbits of the isometries $z^i \rightarrow exp(i\alpha)z^i$ on the sphere S^5: $\sum z^i \bar{z}^i = R^2$. The metric of CP_2 is obtained by projecting the metric of S^5 orthogonally to the orbits of the isometries. Therefore the distance between the points of CP_2 is that between the representative orbits on S^5. The line element has the following form in the complex coordinates

$$ds^2 = g_{a\bar{b}}d\xi^a d\bar{\xi}^b \ , \tag{4}$$

where the Hermitian, in fact Kähler metric $g_{a\bar{b}}$ is defined by

$$g_{a\bar{b}} = R^2 \partial_a \partial_{\bar{b}} K \ , \tag{5}$$

where the function K, Kähler function, is defined as

$$K = log(F) \ ,$$
$$F = 1 + r^2 \ . \tag{6}$$

The representation of the metric is given by

$$\frac{ds^2}{R^2} = \frac{(dr^2 + r^2\sigma_3^2)}{F^2} + \frac{r^2(\sigma_1^2 + \sigma_2^2)}{F} \ , \tag{7}$$

where the quantities σ_i are defined as

$$r^2\sigma_1 = Im(\xi^1 d\xi^2 - \xi^2 d\xi^1) \ ,$$
$$r^2\sigma_2 = -Re(\xi^1 d\xi^2 - \xi^2 d\xi^1) \ ,$$
$$r^2\sigma_3 = -Im(\xi^1 d\bar{\xi}^1 + \xi^2 d\bar{\xi}^2) \ . \tag{8}$$

The vierbein forms, which satisfy the defining relation

$$s_{kl} = R^2 \sum_A e_k^A e_l^A \ , \tag{9}$$

are given by

$$e^0 = \frac{dr}{F} \ , \quad e^1 = \frac{r\sigma_1}{\sqrt{F}} \ ,$$
$$e^2 = \frac{r\sigma_2}{\sqrt{F}} \ , \quad e^3 = \frac{r\sigma_3}{F} \ . \tag{10}$$

The explicit representations of vierbein vectors are given by

$$e^0 = \frac{dr}{F} \ , \qquad\qquad e^1 = \frac{r(sin\Theta cos\Psi d\Phi + sin\Psi d\Theta)}{2\sqrt{F}} \ ,$$
$$e^2 = \frac{r(sin\Theta sin\Psi d\Phi - cos\Psi d\Theta)}{2\sqrt{F}} \ , \quad e^3 = \frac{r(d\Psi + cos\Theta d\Phi)}{2F} \ . \tag{11}$$

The explicit representation of the line element is given by the expression

$$ds^2/R^2 = dr^2/F^2 + (r^2/4F^2)(d\Psi + cos\Theta d\Phi)^2 + (r^2/4F)(d\Theta^2 + sin^2\Theta d\Phi^2) \ . \tag{12}$$

The vierbein connection satisfying the defining relation

$$de^A = -V^A_B \wedge e^B , \tag{13}$$

is given by

$$
\begin{aligned}
V_{01} &= -\frac{e^1}{r} , & V_{23} &= \frac{e^1}{r} , \\
V_{02} &= -\frac{e^2}{r} , & V_{31} &= \frac{e^2}{r} , \\
V_{03} &= (r - \frac{1}{r})e^3 , & V_{12} &= (2r + \frac{1}{r})e^3 .
\end{aligned}
\tag{14}
$$

The representation of the covariantly constant curvature tensor is given by

$$
\begin{aligned}
R_{01} &= e^0 \wedge e^1 - e^2 \wedge e^3 , & R_{23} &= e^0 \wedge e^1 - e^2 \wedge e^3 , \\
R_{02} &= e^0 \wedge e^2 - e^3 \wedge e^1 , & R_{31} &= -e^0 \wedge e^2 + e^3 \wedge e^1 , \\
R_{03} &= 4e^0 \wedge e^3 + 2e^1 \wedge e^2 , & R_{12} &= 2e^0 \wedge e^3 + 4e^1 \wedge e^2 .
\end{aligned}
\tag{15}
$$

Metric defines a real, covariantly constant, and therefore closed 2-form J

$$J = -ig_{a\bar{b}}d\xi^a d\bar{\xi}^b , \tag{16}$$

the so called Kähler form. Kähler form J defines in CP_2 a symplectic structure because it satisfies the condition

$$J^k_{\ r}J^{rl} = -s^{kl} . \tag{17}$$

The form J is integer valued and by its covariant constancy satisfies free Maxwell equations. Hence it can be regarded as a curvature form of a $U(1)$ gauge potential B carrying a magnetic charge of unit $1/2g$ (g denotes the gauge coupling). Locally one has therefore

$$J = dB , \tag{18}$$

where B is the so called Kähler potential, which is not defined globally since J describes homological magnetic monopole.

It should be noticed that the magnetic flux of J through a 2-surface in CP_2 is proportional to its homology equivalence class, which is integer valued. The explicit representations of J and B are given by

$$
\begin{aligned}
B &= 2re^3 , \\
J &= 2(e^0 \wedge e^3 + e^1 \wedge e^2) = \frac{r}{F^2}dr \wedge (d\Psi + cos\Theta d\Phi) + \frac{r^2}{2F}sin\Theta d\Theta d\Phi .
\end{aligned}
\tag{19}
$$

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The vierbein curvature form and Kähler form are covariantly constant and have in the complex coordinates only components of type (1,1).

Useful coordinates for CP_2 are the so called canonical coordinates in which Kähler potential and Kähler form have very simple expressions

$$
\begin{aligned}
B &= \sum_{k=1,2} P_k dQ_k \;, \\
J &= \sum_{k=1,2} dP_k \wedge dQ_k \;.
\end{aligned}
\tag{20}
$$

The relationship of the canonical coordinates to the "spherical" coordinates is given by the equations

$$
\begin{aligned}
P_1 &= -\frac{1}{1+r^2} \;, \\
P_2 &= \frac{r^2 cos\Theta}{2(1+r^2)} \;, \\
Q_1 &= \Psi \;, \\
Q_2 &= \Phi \;.
\end{aligned}
\tag{21}
$$

1.3 Spinors in CP_2

CP_2 doesn't allow spinor structure in the conventional sense [5]. However, the coupling of the spinors to a half odd multiple of the Kähler potential leads to a respectable spinor structure. Because the delicacies associated with the spinor structure of CP_2 play a fundamental role in TGD, the arguments of Hawking are repeated here.

To see how the space can fail to have an ordinary spinor structure consider the parallel transport of the vierbein in a simply connected space M. The parallel propagation around a closed curve with a base point x leads to a rotated vierbein at x: $e^A = R_B^A e^B$ and one can associate to each closed path an element of $SO(4)$.

Consider now a one-parameter family of closed curves $\gamma(v) : v \in (0,1)$ with the same base point x and $\gamma(0)$ and $\gamma(1)$ trivial paths. Clearly these paths define a sphere S^2 in M and the element $R_B^A(v)$ defines a closed path in $SO(4)$. When the sphere S^2 is contractible to a point e.g., homologically trivial, the path in $SO(4)$ is also contractible to a point and therefore represents a trivial element of the homotopy group $\Pi_1(SO(4)) = Z_2$.

For a homologically nontrivial 2-surface S^2 the associated path in $SO(4)$ can be homotopically nontrivial and therefore corresponds to a nonclosed path in the covering group Spin(4) (leading from the matrix 1 to -1 in the matrix representation). Assume this is the case.

Assume now that the space allows spinor structure. Then one can parallel propagate also spinors and by the above construction associate a closed path

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of Spin(4) to the surface S^2. Now, however this path corresponds to a lift of the corresponding $SO(4)$ path and cannot be closed. Thus one ends up with a contradiction.

From the preceding argument it is clear that one could compensate the non-allowed -1- factor associated with the parallel transport of the spinor around the sphere S^2 by coupling it to a gauge potential in such a way that in the parallel transport the gauge potential introduces a compensating -1-factor. For a $U(1)$ gauge potential this factor is given by the exponential $exp(i2\Phi)$, where Φ is the magnetic flux through the surface. This factor has the value -1 provided the $U(1)$ potential carries half odd multiple of Dirac charge $1/2g$. In case of CP_2 the required gauge potential is half odd multiple of the Kähler potential B defined previously. In the case of $M^4 \times CP_2$ one can in addition couple the spinor components with different chiralities independently to an odd multiple of $B/2$.

1.4 Geodesic sub-manifolds of CP_2

Geodesic sub-manifolds are defined as sub-manifolds having common geodesic lines with the imbedding space. As a consequence the second fundamental form of the geodesic manifold vanishes, which means that the tangent vectors h_α^k (understood as vectors of H) are covariantly constant quantities with respect to the covariant derivative taking into account that the tangent vectors are vectors both with respect to H and X^4.

In [3] a general characterization of the geodesic sub-manifolds for an arbitrary symmetric space G/H is given. Geodesic sub-manifolds are in 1-1-correspondence with the so called Lie triple systems of the Lie-algebra g of the group G. The Lie triple system t is defined as a subspace of g characterized by the closedness property with respect to double commutation

$$[X, [Y, Z]] \in t \ \text{ for } \ X, Y, Z \in t \ . \tag{22}$$

$SU(3)$ allows, besides geodesic lines, two nonequivalent (not isometry related) geodesic spheres. This is understood by observing that $SU(3)$ allows two nonequivalent $SU(2)$ algebras corresponding to subgroups $SO(3)$ (orthogonal 3×3 matrices) and the usual isospin group $SU(2)$. By taking any subset of two generators from these algebras, one obtains a Lie triple system and by exponentiating this system, one obtains a 2-dimensional geodesic sub-manifold of CP_2.

Standard representatives for the geodesic spheres of CP_2 are given by the equations

$$S_I^2 \ : \ \xi^1 = \bar{\xi}^2 \ \text{ or equivalently } \ (\Theta = \pi/2, \Psi = 0) \ ,$$

$$S_{II}^2 \ : \ \xi^1 = \xi^2 \ \text{ or equivalently } \ (\Theta = \pi/2, \Phi = 0) \ .$$

The non-equivalence of these sub-manifolds is clear from the fact that isometries act as holomorphic transformations in CP_2. The vanishing of the second fundamental form is also easy to verify. The first geodesic manifold is homologically trivial: in fact, the induced Kähler form vanishes identically for S_I^2. S_{II}^2 is homologically nontrivial and the flux of the Kähler form gives its homology equivalence class.

2 CP_2 geometry and standard model symmetries

2.1 Identification of the electro-weak couplings

The delicacies of the spinor structure of CP_2 make it a unique candidate for space S. First, the coupling of the spinors to the $U(1)$ gauge potential defined by the Kähler structure provides the missing $U(1)$ factor in the gauge group. Secondly, it is possible to couple different H-chiralities independently to a half odd multiple of the Kähler potential. Thus the hopes of obtaining a correct spectrum for the electromagnetic charge are considerable. In the following it will be demonstrated that the couplings of the induced spinor connection are indeed those of the GWS model [4] and in particular that the right handed neutrinos decouple completely from the electro-weak interactions.

To begin with, recall that the space H allows to define three different chiralities for spinors. Spinors with fixed H-chirality $e = \pm 1$, CP_2-chirality l, r and M^4-chirality L, R are defined by the condition

$$
\begin{aligned}
\Gamma \Psi &= e\Psi \ , \\
e &= \pm 1 \ ,
\end{aligned}
\tag{23}
$$

where Γ denotes the matrix $\Gamma_9 = \gamma_5 \times \gamma_5$, $1 \times \gamma_5$ and $\gamma_5 \times 1$ respectively. Clearly, for a fixed H-chirality CP_2- and M^4-chiralities are correlated.

The spinors with H-chirality $e = \pm 1$ can be identified as quark and lepton like spinors respectively. The separate conservation of baryon and lepton numbers can be understood as a consequence of generalized chiral invariance if this identification is accepted. For the spinors with a definite H-chirality one can identify the vielbein group of CP_2 as the electro-weak group: $SO(4) = SU(2)_L \times SU(2)_R$.

The covariant derivatives are defined by the spinorial connection

$$
A = V + \frac{B}{2}(n_+ 1_+ + n_- 1_-) \ .
\tag{24}
$$

Here V and B denote the projections of the vielbein and Kähler gauge potentials respectively and $1_{+(-)}$ projects to the spinor H-chirality $+(-)$. The integers $n_\pm$ are odd from the requirement of a respectable spinor structure.

The explicit representation of the vielbein connection V and of B are given by the equations

$$
\begin{array}{llll}
V_{01} & = & -\frac{e^1}{r} \ , & \qquad V_{23} & = & \frac{e^1}{r} \ , \\
V_{02} & = & -\frac{e^2}{r} \ , & \qquad V_{31} & = & \frac{e^2}{r} \ , \\
V_{03} & = & (r - \frac{1}{r})e^3 \ , & \qquad V_{12} & = & (2r + \frac{1}{r})e^3 \ ,
\end{array}
\tag{25}
$$

and

$$
B \ = \ 2re^3 \ , \tag{26}
$$

respectively. The explicit representation of the vielbein is not needed here.

Let us first show that the charged part of the spinor connection couples purely left handedly. Identifying Σ_3^0 and Σ_2^1 as the diagonal (neutral) Lie-algebra generators of $SO(4)$, one finds that the charged part of the spinor connection is given by

$$
A_{ch} \ = \ 2V_{23}I_L^1 + 2V_{13}I_L^2 \ , \tag{27}
$$

where one have defined

$$
\begin{array}{lll}
I_L^1 & = & \dfrac{(\Sigma_{01} - \Sigma_{23})}{2} \ , \\[2mm]
I_L^2 & = & \dfrac{(\Sigma_{02} - \Sigma_{13})}{2} \ .
\end{array}
\tag{28}
$$

A_{ch} is clearly left handed so that one can perform the identification

$$
W^\pm \ = \ \frac{2(e^1 \pm ie^2)}{r} \ , \tag{29}
$$

where $W^\pm$ denotes the charged intermediate vector boson.

Consider next the identification of the neutral gauge bosons γ and Z^0 as appropriate linear combinations of the two functionally independent quantities

$$
\begin{array}{lll}
X & = & re^3 \ , \\[2mm]
Y & = & \dfrac{e^3}{r} \ ,
\end{array}
\tag{30}
$$

appearing in the neutral part of the spinor connection. We show first that the mere requirement that photon couples vectorially implies the basic coupling structure of the GWS model leaving only the value of Weinberg angle undetermined.

To begin with let us define

$$\begin{aligned}
\bar{\gamma} &= aX + bY \;, \\
\bar{Z}^0 &= cX + dY \;,
\end{aligned} \tag{31}$$

where the normalization condition

$$ad - bc = 1 \;,$$

is satisfied. The physical fields γ and Z^0 are related to $\bar{\gamma}$ and $\bar{Z}^0$ by simple normalization factors.

Expressing the neutral part of the spinor connection in term of these fields one obtains

$$\begin{aligned}
A_{nc} &= [(c+d)2\Sigma_{03} + (2d-c)2\Sigma_{12} + d(n_+ 1_+ + n_- 1_-)]\bar{\gamma} \\
&+ [(a-b)2\Sigma_{03} + (a-2b)2\Sigma_{12} - b(n_+ 1_+ + n_- 1_-)]\bar{Z}^0 \;.
\end{aligned} \tag{32}$$

Identifying Σ_{12} and $\Sigma_{03} = 1 \times \gamma_5 \Sigma_{12}$ as vectorial and axial Lie-algebra generators, respectively, the requirement that γ couples vectorially leads to the condition

$$c = -d \;. \tag{33}$$

Using this result plus previous equations, one obtains for the neutral part of the connection the expression

$$A_{nc} = \gamma Q_{em} + Z^0 (I_L^3 - sin^2\theta_W Q_{em}) \;. \tag{34}$$

Here the electromagnetic charge Q_{em} and the weak isospin are defined by

$$\begin{aligned}
Q_{em} &= \Sigma^{12} + \frac{(n_+ 1_+ + n_- 1_-)}{6} \;, \\
I_L^3 &= \frac{(\Sigma^{12} - \Sigma^{03})}{2} \;.
\end{aligned} \tag{35}$$

The fields γ and Z^0 are defined via the relations

$$\begin{aligned}
\gamma &= 6d\bar{\gamma} = \frac{6}{(a+b)}(aX + bY) \;, \\
Z^0 &= 4(a+b)\bar{Z}^0 = 4(X - Y) \;.
\end{aligned} \tag{36}$$

The value of the Weinberg angle is given by

9

$$sin^2\theta_W = \frac{3b}{2(a+b)} \ , \tag{37}$$

and is not fixed completely. Observe that right handed neutrinos decouple completely from the electro-weak interactions.

The determination of the value of Weinberg angle is a dynamical problem. The angle is completely fixed once the YM action is fixed by requiring that action contains no cross term of type γZ^0. Pure symmetry non-broken electro-weak YM action leads to a definite value for the Weinberg angle. One can however add a symmetry breaking term proportional to Kähler action and this changes the value of the Weinberg angle.

To evaluate the value of the Weinberg angle one can express the neutral part F_{nc} of the induced gauge field as

$$F_{nc} = 2R_{03}\Sigma^{03} + 2R_{12}\Sigma^{12} + J(n_+1_+ + n_-1_-) \ , \tag{38}$$

where one has

$$\begin{aligned}
R_{03} &= 2(2e^0 \wedge e^3 + e^1 \wedge e^2) \ , \\
R_{12} &= 2(e^0 \wedge e^3 + 2e^1 \wedge e^2) \ , \\
J &= 2(e^0 \wedge e^3 + e^1 \wedge e^2) \ ,
\end{aligned} \tag{39}$$

in terms of the fields γ and Z^0 (photon and Z- boson)

$$F_{nc} = \gamma Q_{em} + Z^0(I_L^3 - sin^2\theta_W Q_{em}) \ . \tag{40}$$

Evaluating the expressions above one obtains for γ and Z^0 the expressions

$$\begin{aligned}
\gamma &= 3J - sin^2\theta_W R_{03} \ , \\
Z^0 &= 2R_{03} \ .
\end{aligned} \tag{41}$$

For the Kähler field one obtains

$$J = \frac{1}{3}(\gamma + sin^2\theta_W Z^0) \ . \tag{42}$$

Expressing the neutral part of the symmetry broken YM action

$$\begin{aligned}
L_{ew} &= L_{sym} + fJ^{\alpha\beta}J_{\alpha\beta} \ , \\
L_{sym} &= \frac{1}{4g^2}Tr(F^{\alpha\beta}F_{\alpha\beta}) \ ,
\end{aligned} \tag{43}$$

where the trace is taken in spinor representation, in terms of γ and Z^0 one obtains for the coefficient X of the γZ^0 cross term (this coefficient must vanish) the expression

$$
\begin{aligned}
X &= -\frac{K}{2g^2} + \frac{fp}{18} \ , \\
K &= Tr\left[Q_{em}(I_L^3 - sin^2\theta_W Q_{em})\right] \ ,
\end{aligned}
\tag{44}
$$

In the general case the value of the coefficient K is given by

$$
K = \sum_i \left[-\frac{(18 + 2n_i^2)sin^2\theta_W}{9}\right] \ ,
\tag{45}
$$

where the sum is over the spinor chiralities, which appear as elementary fermions and n_i is the integer describing the coupling of the spinor field to the Kähler potential. The cross term vanishes provided the value of the Weinberg angle is given by

$$
sin^2\theta_W = \frac{9\sum_i 1}{(fg^2 + 2\sum_i(18 + n_i^2))} \ .
\tag{46}
$$

In the scenario where both leptons and quarks are elementary fermions the value of the Weinberg angle is given by

$$
sin^2\theta_W = \frac{9}{(\frac{fg^2}{2} + 28)} \ .
\tag{47}
$$

The bare value of the Weinberg angle is $9/28$ in this scenario, which is quite close to the typical value $9/24$ of GUTs [6].

2.2 Discrete symmetries

The treatment of discrete symmetries C, P, and T is based on the following requirements:
a) Symmetries must be realized as purely geometric transformations.
b) Transformation properties of the field variables should be essentially the same as in the conventional quantum field theories [1].

The action of the reflection P on spinors of is given by

$$
\Psi \rightarrow P\Psi = \gamma^0 \times \gamma^0\Psi \ .
\tag{48}
$$

in the representation of the gamma matrices for which γ^0 is diagonal. It should be noticed that W and Z^0 bosons break parity symmetry as they should since their charge matrices do not commute with the matrix of P.

The guess that a complex conjugation in CP_2 is associated with T transformation of the physicist turns out to be correct. One can verify by a direct calculation that pure Dirac action is invariant under T realized according to

$$
\begin{aligned}
m^k &\rightarrow T(M^k) \ , \\
\xi^k &\rightarrow \bar{\xi}^k \ , \\
\Psi &\rightarrow \gamma^1 \gamma^3 \times 1 \times \Psi \ .
\end{aligned}
\tag{49}
$$

The operation bearing closest resemblance to the ordinary charge conjugation corresponds geometrically to complex conjugation in CP_2:

$$
\begin{aligned}
\xi^k &\rightarrow \bar{\xi}^k \ , \\
\Psi &\rightarrow \gamma^2 \gamma^0 \times 1 \times \Psi^\dagger \ .
\end{aligned}
\tag{50}
$$

As one might have expected symmetries CP and T are exact symmetries of the pure Dirac action.

3 Basic facts about induced gauge fields

Since the classical gauge fields are closely related in TGD framework, it is not possible to have space-time sheets carrying only single kind of gauge field. For instance, em fields are accompanied by Z^0 fields for extremals of Kähler action. Weak forces is however absent unless the space-time sheets contains topologically condensed exotic weakly charged particles responding to this force. Same applies to classical color forces. The fact that these long range fields are present forces to assume that there exists a hierarchy of scaled up variants of standard model physics identifiable in terms of dark matter.

Classical em fields are always accompanied by Z^0 field and some components of color gauge field. For extremals having homologically non-trivial sphere as a CP_2 projection em and Z^0 fields are the only non-vanishing electroweak gauge fields. For homologically trivial sphere only W fields are non-vanishing. Color rotations does not affect the situation.

For vacuum extremals all electro-weak gauge fields are in general non-vanishing although the net gauge field has U(1) holonomy by 2-dimensionality of the CP_2 projection. Color gauge field has $U(1)$ holonomy for all space-time surfaces and quantum classical correspondence suggest a weak form of color confinement meaning that physical states correspond to color neutral members of color multiplets.

3.1 Induced gauge fields for space-times for which CP_2 projection is a geodesic sphere

If one requires that space-time surface is an extremal of Kähler action and has a 2-dimensional CP_2 projection, only vacuum extremals and space-time surfaces

for which CP$_2$ projection is a geodesic sphere, are allowed. Homologically non-trivial geodesic sphere correspond to vanishing W fields and homologically non-trivial sphere to non-vanishing W fields but vanishing γ and Z^0. This can be verified by explicit examples.

$r = \infty$ surface gives rise to a homologically non-trivial geodesic sphere for which e_0 and e_3 vanish imply the vanishing of W field. For space-time sheets for which CP$_2$ projection is $r = \infty$ homologically non-trivial geodesic sphere of CP_2 one has

$$\gamma = (\frac{3}{4} - \frac{sin^2(\theta_W)}{2})Z^0 \simeq \frac{5Z^0}{8} \ .$$

The induced W fields vanish in this case and they vanish also for all geodesic sphere obtained by $SU(3)$ rotation.

$Im(\xi^1) = Im(\xi^2) = 0$ corresponds to homologically trivial geodesic sphere. A more general representative is obtained by using for the phase angles of standard complex CP_2 coordinates constant values. In this case e^1 and e^3 vanish so that the induced em, Z^0, and Kähler fields vanish but induced W fields are non-vanishing. This holds also for surfaces obtained by color rotation. Hence one can say that for non-vacuum extremals with 2-D CP$_2$ projection color rotations and weak symmetries commute.

3.2 Space-time surfaces with vanishing em, Z^0, or Kähler fields

In the following the induced gauge fields are studied for general space-time surface without assuming the extremal property. In fact, extremal property reduces the study to the study of vacuum extremals and surfaces having geodesic sphere as a CP_2 projection and in this sense the following arguments are somewhat obsolete in their generality.

3.2.1 Space-times with vanishing em, Z^0, or Kähler fields

The following considerations apply to a more general situation in which the homologically trivial geodesic sphere and extremal property are not assumed. It must be emphasized that this case is possible in TGD framework only for a vanishing Kähler field.

Using spherical coordinates (r, Θ, Ψ, Φ) for CP_2, the expression of Kähler form reads as

$$
\begin{aligned}
J &= \frac{r}{F^2}dr \wedge (d\Psi + cos(\Theta)d\Phi) + \frac{r^2}{2F}sin(\Theta)d\Theta \wedge d\Phi \ , \\
F &= 1 + r^2 \ .
\end{aligned}
\tag{51}
$$

The general expression of electromagnetic field reads as

$$\begin{aligned}
F_{em} &= (3+2p)\frac{r}{F^2}dr \wedge (d\Psi + cos(\Theta)d\Phi) + (3+p)\frac{r^2}{2F}sin(\Theta)d\Theta \wedge d\Phi , \\
p &= sin^2(\Theta_W) ,
\end{aligned} \tag{52}$$

where Θ_W denotes Weinberg angle.

a) The vanishing of the electromagnetic fields is guaranteed, when the conditions

$$\begin{aligned}
\Psi &= k\Phi , \\
(3+2p)\frac{1}{r^2 F}(d(r^2)/d\Theta)(k + cos(\Theta)) &+ (3+p)sin(\Theta) = 0 ,
\end{aligned} \tag{53}$$

hold true. The conditions imply that CP_2 projection of the electromagnetically neutral space-time is 2-dimensional. Solving the differential equation one obtains

$$\begin{aligned}
r &= \sqrt{\frac{X}{1-X}} , \\
X &= D\left[|\frac{(k+u)}{C}|\right]^\epsilon , \\
u &\equiv cos(\Theta) , \quad C = k + cos(\Theta_0) , \quad D = \frac{r_0^2}{1+r_0^2} , \quad \epsilon = \frac{3+p}{3+2p} ,
\end{aligned} \tag{54}$$

where C and D are integration constants. $0 \leq X \leq 1$ is required by the reality of r. $r = 0$ would correspond to $X = 0$ giving $u = -k$ achieved only for $|k| \leq 1$ and $r = \infty$ to $X = 1$ giving $|u + k| = [(1+r_0^2)/r_0^2]^{(3+2p)/(3+p)}$ achieved only for

$$sign(u+k) \times [\frac{1+r_0^2}{r_0^2}]^{\frac{3+2p}{3+p}} \leq k+1 ,$$

where $sign(x)$ denotes the sign of x.

The expressions for Kähler form and Z^0 field are given by

$$\begin{aligned}
J &= -\frac{p}{3+2p}X du \wedge d\Phi , \\
Z^0 &= -\frac{6}{p}J .
\end{aligned} \tag{55}$$

The components of the electromagnetic field generated by varying vacuum parameters are proportional to the components of the Kähler field: in particular, the magnetic field is parallel to the Kähler magnetic field. The generation of a long range Z^0 vacuum field is a purely TGD based feature not encountered in the standard gauge theories.

b) The vanishing of Z^0 fields is achieved by the replacement of the parameter ϵ with $\epsilon = 1/2$ as becomes clear by considering the condition stating that Z^0 field vanishes identically. Also the relationship $F_{em} = 3J = -\frac{3}{4}\frac{r^2}{F}du \wedge d\Phi$ is useful.

c) The vanishing Kähler field corresponds to $\epsilon = 1, p = 0$ in the formula for em neutral space-times. In this case classical em and Z^0 fields are proportional to each other:

$$
\begin{aligned}
Z^0 &= 2e^0 \wedge e^3 = \frac{r}{F^2}(k + u)\frac{\partial r}{\partial u}du \wedge d\Phi = (k + u)du \wedge d\Phi \ , \\
r &= \sqrt{\frac{X}{1 - X}} \ , \quad X = D|k + u| \ , \\
\gamma &= -\frac{p}{2}Z^0 \ .
\end{aligned}
\tag{56}
$$

For a vanishing value of Weinberg angle ($p = 0$) em field vanishes and only Z^0 field remains as a long range gauge field. Vacuum extremals for which long range Z^0 field vanishes but em field is non-vanishing are not possible.

3.2.2 The effective form of CP_2 metric for surfaces with 2-dimensional CP_2 projection

The effective form of the CP_2 metric for a space-time having vanishing em, Z^0, or Kähler field is of practical value in the case of vacuum extremals and is given by

$$
\begin{aligned}
ds^2_{eff} &= (s_{rr}(\frac{dr}{d\Theta})^2 + s_{\Theta\Theta})d\Theta^2 + (s_{\Phi\Phi} + 2ks_{\Phi\Psi})d\Phi^2 = \frac{R^2}{4}[s^{eff}_{\Theta\Theta}d\Theta^2 + s^{eff}_{\Phi\Phi}d\Phi^2] \ , \\
s^{eff}_{\Theta\Theta} &= X \times \left[\frac{\epsilon^2(1 - u^2)}{(k + u)^2} \times \frac{1}{1 - X} + 1 - X\right] \ , \\
s^{eff}_{\Phi\Phi} &= X \times \left[(1 - X)(k + u)^2 + 1 - u^2\right] \ ,
\end{aligned}
\tag{57}
$$

and is useful in the construction of vacuum imbedding of, say Schwartchild metric.

3.2.3 Topological quantum numbers

Space-times for which either em, Z^0, or Kähler field vanishes decompose into regions characterized by six vacuum parameters: two of these quantum numbers (ω_1 and ω_2) are frequency type parameters, two (k_1 and k_2) are wave vector like quantum numbers, two of the quantum numbers (n_1 and n_2) are integers. The parameters ω_i and n_i will be referred as electric and magnetic quantum numbers. The existence of these quantum numbers is not a feature of these solutions alone but represents a much more general phenomenon differentiating in a clear cut manner between TGD and Maxwell's electrodynamics.

The simplest manner to avoid surface Kähler charges and discontinuities or infinities in the derivatives of CP_2 coordinates on the common boundary of two neighboring regions with different vacuum quantum numbers is topological field quantization, 3-space decomposes into disjoint topological field quanta, 3-surfaces having outer boundaries with possibly macroscopic size.

Under rather general conditions the coordinates Ψ and Φ can be written in the form

$$
\begin{aligned}
\Psi &= \omega_2 m^0 + k_2 m^3 + n_2 \phi + \text{Fourier expansion} \ , \\
\Phi &= \omega_1 m^0 + k_1 m^3 + n_1 \phi + \text{Fourier expansion} \ .
\end{aligned}
\tag{58}
$$

m^0, m^3 and ϕ denote the coordinate variables of the cylindrical M^4 coordinates) so that one has $k = \omega_2/\omega_1 = n_2/n_1 = k_2/k_1$. The regions of the space-time surface with given values of the vacuum parameters ω_i, k_i and n_i and m and C are bounded by the surfaces at which space-time surface becomes ill-defined, say by $r > 0$ or $r < \infty$ surfaces.

The space-time surface decomposes into regions characterized by different values of the vacuum parameters r_0 and Θ_0. At $r = \infty$ surfaces n_2, ω_2 and m can change since all values of Ψ correspond to the same point of CP_2: at $r = 0$ surfaces also n_1 and ω_1 can change since all values of Φ correspond to same point of CP_2, too. If $r = 0$ or $r = \infty$ is not in the allowed range space-time surface develops a boundary.

This implies what might be called topological quantization since in general it is not possible to find a smooth global imbedding for, say a constant magnetic field. Although global imbedding exists it decomposes into regions with different values of the vacuum parameters and the coordinate u in general possesses discontinuous derivative at $r = 0$ and $r = \infty$ surfaces. A possible manner to avoid edges of space-time is to allow field quantization so that 3-space (and field) decomposes into disjoint quanta, which can be regarded as structurally stable units a 3-space (and of the gauge field). This doesn't exclude partial join along boundaries for neighboring field quanta provided some additional conditions guaranteing the absence of edges are satisfied.

For instance, the vanishing of the electromagnetic fields implies that the condition

$$
\Omega \equiv \frac{\omega_2}{n_2} - \frac{\omega_1}{n_1} = 0 \ ,
\tag{59}
$$

is satisfied. In particular, the ratio ω_2/ω_1 is rational number for the electromagnetically neutral regions of space-time surface. The change of the parameter n_1 and n_2 (ω_1 and ω_2) in general generates magnetic field and therefore these integers will be referred to as magnetic (electric) quantum numbers.

4 p-Adic numbers and TGD

4.1 p-Adic number fields

p-Adic numbers (p is prime: 2,3,5,...) can be regarded as a completion of the rational numbers using a norm, which is different from the ordinary norm of real numbers [8]. p-Adic numbers are representable as power expansion of the prime number p of form:

$$x \;=\; \sum_{k \geq k_0} x(k)p^k, \; x(k) = 0, ..., p-1 \; . \tag{60}$$

The norm of a p-adic number is given by

$$|x| \;=\; p^{-k_0(x)} \; . \tag{61}$$

Here $k_0(x)$ is the lowest power in the expansion of the p-adic number. The norm differs drastically from the norm of the ordinary real numbers since it depends on the lowest pinary digit of the p-adic number only. Arbitrarily high powers in the expansion are possible since the norm of the p-adic number is finite also for numbers, which are infinite with respect to the ordinary norm. A convenient representation for p-adic numbers is in the form

$$x \;=\; p^{k_0}\varepsilon(x) \; , \tag{62}$$

where $\varepsilon(x) = k + ...$ with $0 < k < p$, is p-adic number with unit norm and analogous to the phase factor $exp(i\phi)$ of a complex number.

The distance function $d(x,y) = |x-y|_p$ defined by the p-adic norm possesses a very general property called ultra-metricity:

$$d(x,z) \;\leq\; max\{d(x,y), d(y,z)\} \; . \tag{63}$$

The properties of the distance function make it possible to decompose R_p into a union of disjoint sets using the criterion that x and y belong to same class if the distance between x and y satisfies the condition

$$d(x,y) \;\leq\; D \; . \tag{64}$$

This division of the metric space into classes has following properties:

a) Distances between the members of two different classes X and Y do not depend on the choice of points x and y inside classes. One can therefore speak about distance function between classes.

b) Distances of points x and y inside single class are smaller than distances between different classes.

c) Classes form a hierarchical tree.

Notice that the concept of the ultra-metricity emerged in physics from the models for spin glasses and is believed to have also applications in biology [10]. The emergence of p-adic topology as the topology of the effective space-time would make ultra-metricity property basic feature of physics.

4.2 Canonical correspondence between p-adic and real numbers

The basic challenge encountered by p-adic physicist is how to map the predictions of the p-adic physics to real numbers. p-Adic probabilities provide a basic example in this respect. Identification via common rationals and canonical identification and its variants have turned out to play a key role in this respect.

4.2.1 Basic form of canonical identification

There exists a natural continuous map $I : R_p \to R_+$ from p-adic numbers to non-negative real numbers given by the "pinary" expansion of the real number for $x \in R$ and $y \in R_p$ this correspondence reads

$$
\begin{aligned}
y &= \sum_{k>N} y_k p^k \to x = \sum_{k<N} y_k p^{-k} \quad , \\
y_k &\in \{0, 1, .., p-1\} \quad .
\end{aligned}
\tag{65}
$$

This map is continuous as one easily finds out. There is however a little difficulty associated with the definition of the inverse map since the pinary expansion like also decimal expansion is not unique $(1 = 0.999...)$ for the real numbers x, which allow pinary expansion with finite number of pinary digits

$$
\begin{aligned}
x &= \sum_{k=N_0}^{N} x_k p^{-k} \quad , \\
x &= \sum_{k=N_0}^{N-1} x_k p^{-k} + (x_N - 1)p^{-N} + (p-1)p^{-N-1} \sum_{k=0,..} p^{-k} \quad .
\end{aligned}
\tag{66}
$$

The p-adic images associated with these expansions are different

$$
y_1 = \sum_{k=N_0}^{N} x_k p^k \quad ,
$$

$$y_2 = \sum_{k=N_0}^{N-1} x_k p^k + (x_N - 1)p^N + (p-1)p^{N+1} \sum_{k=0,..} p^k$$
$$= y_1 + (x_N - 1)p^N - p^{N+1} \ , \tag{67}$$

so that the inverse map is either two-valued for p-adic numbers having expansion with finite pinary digits or single valued and discontinuous and non-surjective if one makes pinary expansion unique by choosing the one with finite pinary digits. The finite pinary digit expansion is a natural choice since in the numerical work one always must use a pinary cutoff on the real axis.

4.2.2 The topology induced by canonical identification

The topology induced by the canonical identification in the set of positive real numbers differs from the ordinary topology. The difference is easily understood by interpreting the p-adic norm as a norm in the set of the real numbers. The norm is constant in each interval $[p^k, p^{k+1})$ (see Fig. 4.2.2) and is equal to the usual real norm at the points $x = p^k$: the usual linear norm is replaced with a piecewise constant norm. This means that p-adic topology is coarser than the usual real topology and the higher the value of p is, the coarser the resulting topology is above a given length scale. This hierarchical ordering of the p-adic topologies will be a central feature as far as the proposed applications of the p-adic numbers are considered.

Ordinary continuity implies p-adic continuity since the norm induced from the p-adic topology is rougher than the ordinary norm. p-Adic continuity implies ordinary continuity from right as is clear already from the properties of the p-adic norm (the graph of the norm is indeed continuous from right). This feature is one clear signature of the p-adic topology.
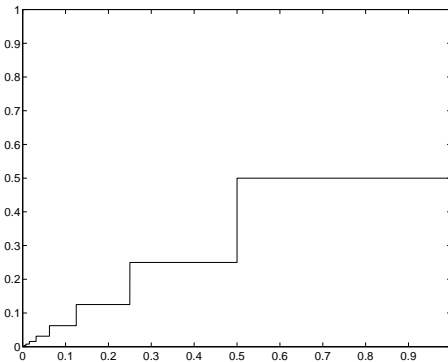

Figure 1: The real norm induced by canonical identification from 2-adic norm.

The linear structure of the p-adic numbers induces a corresponding structure in the set of the non-negative real numbers and p-adic linearity in general differs from the ordinary concept of linearity. For example, p-adic sum is equal to real sum only provided the summands have no common pinary digits. Furthermore, the condition $x +_p y < max\{x, y\}$ holds in general for the p-adic sum of the real numbers. p-Adic multiplication is equivalent with the ordinary multiplication only provided that either of the members of the product is power of p. Moreover one has $x \times_p y < x \times y$ in general. The p-Adic negative -1_p associated with p-adic unit 1 is given by $(-1)_p = \sum_k (p-1)p^k$ and defines p-adic negative for each real number x. An interesting possibility is that p-adic linearity might replace the ordinary linearity in some strongly nonlinear systems so these systems would look simple in the p-adic topology.

These results suggest that canonical identification is involved with some deeper mathematical structure. The following inequalities hold true:

$$
\begin{aligned}
(x + y)_R &\leq x_R + y_R \ , \\
|x|_p|y|_R \leq (xy)_R &\leq x_R y_R \ ,
\end{aligned}
\tag{68}
$$

where $|x|_p$ denotes p-adic norm. These inequalities can be generalized to the case of $(R_p)^n$ (a linear vector space over the p-adic numbers).

$$
\begin{aligned}
(x + y)_R &\leq x_R + y_R \ , \\
|\lambda|_p|y|_R \leq (\lambda y)_R &\leq \lambda_R y_R \ ,
\end{aligned}
\tag{69}
$$

where the norm of the vector $x \in T_p^n$ is defined in some manner. The case of Euclidian space suggests the definition

$$
(x_R)^2 = (\sum_n x_n^2)_R \ .
\tag{70}
$$

These inequalities resemble those satisfied by the vector norm. The only difference is the failure of linearity in the sense that the norm of a scaled vector is not obtained by scaling the norm of the original vector. Ordinary situation prevails only if the scaling corresponds to a power of p.

These observations suggests that the concept of a normed space or Banach space might have a generalization and physically the generalization might apply to the description of some non-linear systems. The nonlinearity would be concentrated in the nonlinear behavior of the norm under scaling.

4.2.3 Modified form of the canonical identification

The original form of the canonical identification is continuous but does not respect symmetries even approximately. This led to a search of variants which would do better in this respect. The modification of the canonical identification applying to rationals only and given by

$$I_Q(q = p^k \times \frac{r}{s}) = p^k \times \frac{I(r)}{I(s)} \tag{71}$$

is uniquely defined for rationals, maps rationals to rationals, has also a symmetry under exchange of target and domain. This map reduces to a direct identification of rationals for $0 \le r < p$ and $0 \le s < p$. It has turned out that it is this map which most naturally appears in the applications. The map is obviously continuous locally since p-adically small modifications of r and s mean small modifications of the real counterparts.

Canonical identification is in a key role in the successful predictions of the elementary particle masses. The predictions for the light elementary particle masses are within extreme accuracy same for I and I_Q but I_Q is theoretically preferred since the real probabilities obtained from p-adic ones by I_Q sum up to one in p-adic thermodynamics.

4.2.4 Generalization of number concept and notion of imbedding space

TGD forces an extension of number concept: roughly a fusion of reals and various p-adic number fields along common rationals is in question. This induces a similar fusion of real and p-adic imbedding spaces. Since finite p-adic numbers correspond always to non-negative reals n-dimensional space R^n must be covered by 2^n copies of the p-adic variant R_p^n of R^n each of which projects to a copy of R_+^n (four quadrants in the case of plane). The common points of p-adic and real imbedding spaces are rational points and most p-adic points are at real infinity.

For a given p-adic space-time sheet most points are literally infinite as real points and the projection to the real imbedding space consists of a discrete set of rational points: the interpretation in terms of the unavoidable discreteness of the physical representations of cognition is natural. Purely local p-adic physics implies real p-adic fractality and thus long range correlations for the real space-time surfaces having enough common points with this projection.

p-Adic fractality means that M^4 projections for the rational points of space-time surface X^4 are related by a direct identification whereas CP_2 coordinates of X^4 at these points are related by I, I_Q or some of its variants implying long range correlates for CP_2 coordinates. Since only a discrete set of points are related in this manner, both real and p-adic field equations can be satisfied and there are no problems with symmetries. p-Adic effective topology is expected to be a good approximation only within some length scale range which means infrared and UV cutoffs. Also multi-p-fractality is possible.

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